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On categoricity of atomic AEC

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Abstract

In recent years, the results about atomic abstract elementary class were summarized by J.T.Baldin [1]. In that book, categoricity problem of atomic AEC is discussed mainly under the assumption of atomic ω -stability (or \ast -excellence). I tried the argument around the problem under some weaker conditions.

1. Atomic AEC and splitting

We recall some definitions.

Definition 1 A class of structures $(\mathbf{K}, \prec_{\mathbf{K}})$ (of a language L) is an *abstract elementary class* (AEC) if the class \mathbf{K} and class of pairs satisfying the binary relation $\prec_{\mathbf{K}}$ are each closed under isomorphism and satisfy the following conditions ;

A1. If $M \prec_{\mathbf{K}} N$, then $M \subseteq N$.

A2. $\prec_{\mathbf{K}}$ is a partial order on \mathbf{K} .

A3. If $\{A_i : i < \delta\}$ is a $\prec_{\mathbf{K}}$ -increasing chain :

(1) $\bigcup_{i < \delta} A_i \in \mathbf{K}$

(2) for each $j < \delta$, $A_j \prec_{\mathbf{K}} \bigcup_{i < \delta} A_i$

(3) if each $A_i \prec_{\mathbf{K}} M \in \mathbf{K}$, then $\bigcup_{i < \delta} A_i \prec_{\mathbf{K}} M$.

A4. If $A, B, C \in \mathbf{K}$, $A \prec_{\mathbf{K}} C$, $B \prec_{\mathbf{K}} C$ and $A \subseteq B$, then $A \prec_{\mathbf{K}} B$.

A5. There is a Löwenheim-Skolem number $LS(\mathbf{K})$ such that if $A \subseteq B \in \mathbf{K}$, there is an $A' \in \mathbf{K}$ with $A \subseteq A' \prec_{\mathbf{K}} B$ and $|A'| \leq |A| + LS(\mathbf{K})$.

Definition 2 We say an AEC $(\mathbf{K}, \prec_{\mathbf{K}})$ is *atomic* if \mathbf{K} is the class of atomic models of a countable complete first order theory and $\prec_{\mathbf{K}}$ is first order elementary submodel.

In the following, \mathbf{K} denotes an atomic AEC.

Definition 3 Let T be a countable first order theory.

A set A contained in a model M of T is *atomic* if every finite sequence in

A realizes a principal type over the empty set.

Let A be an atomic set.

$S_{at}(A)$ is the collection of $p \in S(A)$ such that if $a \in \mathcal{M}$ realizes p , Aa is atomic (where \mathcal{M} is the big model).

We refer to a $p \in S_{at}(A)$ as an *atomic type*.

We consider the notion of stability for atomic types.

Definition 4 The atomic class \mathbf{K} is λ – stable if for every $M \in \mathbf{K}$ of cardinality λ , $|S_{at}(M)| = \lambda$.

Example 5 ([1]) 1. Let \mathbf{K}_1 be the class of atomic models of the theory of dense linear order without endpoints. Then \mathbf{K}_1 is not ω –stable.

2. Let \mathbf{K}_2 be the class of atomic models of the theory of the ordered Abelian group of rationals. Then \mathbf{K}_2 is ω –stable.

The notion of independence by splitting is available in this context.

Definition 6 A complete type p over B splits over $A \subset B$ if there are $b, c \in B$ which realize the same type over A and a formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.

Let A, B, C be atomic.

We write $A \downarrow_C B$ and say A is independent from B over C if for any finite sequence $a \in A$, $\text{tp}_{at}(a/B)$ does not split over some finite subset of C .

Fact 7 ([1]) Under the atomic ω –stable assumption of $(\mathbf{K}, \prec_{\mathbf{K}})$ (and some assumption of parameters), the independence relation by splitting (over models) satisfies almost all forking axioms.

Theorem 8 ([1]) If \mathbf{K} is ω –stable and has a model of power \aleph_1 , then it has a model of power \aleph_2 .

2. Atomic AEC without infinite splitting chain

In Baldwin’s book [1] they argue the categoricity of atomic AEC under ω –stability assumption of atomic types. I considered the same problem under some weaker conditions.

Definition 9 Let \mathbf{K} be an atomic AEC and $M \in \mathbf{K}$.

M has no infinite splitting chain if for any nonalgebraic $p \in S_{at}(M)$, there is no increasing sequence $\{A_i\}_{i < \omega} (\subset M)$ such that $p \upharpoonright A_{i+1}$ splits over A_i for all $i < \omega$.

We can prove the next facts.

Fact 10 *If \mathbf{K} is ω -stable, then no model of \mathbf{K} has infinite splitting chain.*

Fact 11 *Under the assumption that $(\mathbf{K}, \prec_{\mathbf{K}})$ has no infinite splitting chain, the independence relation by splitting (over models) satisfies almost all forking axioms.*

3. Existence of pregeometry

In [1], categoricity of atomic AEC are proved by means of the fact that every model is prime and minimal over a basis of some pregeometry given by a quasi-minimal set. So I tried to define pregeometry in the present context.

At first we prove the next proposition which is some modification of Theorem 8 above.

Proposition 12 *If there are $N \in \mathbf{K}$ with $|N| > \aleph_0$ and a nonalgebraic type $p(x) \in S_{at}^1(N)$ such that N has no infinite splitting chain.*

Then there are $M \in \mathbf{K}$ with $|M| = \aleph_2$ and a nonalgebraic type $q(x) \in S_{at}^1(M)$ such that M has no infinite splitting chain and q does not split over some $b \in M$, and $q \upharpoonright b$ has a Morley sequence I in M with $|I| = \aleph_2$.

Moreover if $|N| = \aleph_1$, then we can take M such that $N \prec M$.

In this note, Morley sequence means the sequence constructed by non-splitting extensions. Thus Morley sequences are indiscernible.

Lemma 13 *Let $M \in \mathbf{K}$ and $p(x) \in S_{at}(M)$.*

Suppose that M has no infinite splitting chain and p does not split over some $b \in M$.

And let $I = \{a_i : i < \alpha\}$ be a Morley sequence of $p \upharpoonright b$ in M .

Then I is totally indiscernible.

In [8], they characterized generically stable types. We try to modify the notion in this context.

Definition 14 *Let $M \in \mathbf{K}$.*

A nonalgebraic type $p(x) \in S_{at}(M)$ is generically stable in M if for some $A \subset M$, p does not split over A and if $I = \{a_i : i < \alpha\}$ is a Morley sequence of $p \upharpoonright A$ in M , then for any $\phi(x) \in L(M)$ -formula, $\{i : M \models \phi(a_i)\}$ is either finite or co-finite.

We can prove the next lemma.

Lemma 15 *Let $M \in \mathbf{K}$ and $q(x) \in S_{at}^1(M)$ be in Proposition 12.*

Then q is generically stable in M .

Moreover if q does not split over b , then q is definable over b and $q \upharpoonright b$ is stationary w.r.t. nonsplitting extension.

We recall the definition of pregeometry.

Definition 16 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X . If the function cl satisfies the following properties, we say (X, cl) is *pregeometry*.

- (I) $A \subset B \implies A \subset \text{cl}(A) \subset \text{cl}(B)$,
- (II) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$,
- (III) (Finite character) $b \in \text{cl}(A) \implies b \in \text{cl}(A_0)$ for some finite $A_0 \subset A$,
- (IV) (Exchange axiom)
 $b \in \text{cl}(A \cup \{c\}) - \text{cl}(A) \implies c \in \text{cl}(A \cup \{b\})$.

We define big type which is a modified notion in [1].

Definition 17 Let $a \in M$ and $A \subset M \in \mathbf{K}$.

A nonalgebraic atomic type $\text{tp}_{\text{at}}(a/A)$ is *big* if there is an atomic model $N \in \mathbf{K}$ such that $A \subset N$ and $\text{tp}_{\text{at}}(a/A)$ has a nonalgebraic atomic extension over N .

In the following we argue under the existence of uncountable model $M \in \mathbf{K}$ and a nonalgebraic type $p(x) \in S_{\text{at}}^1(M)$. We may assume that p has what is called a minimal U-rank, or U-rank = 1.

Lemma 18 Let \mathbf{K} has no infinite splitting chain and $M \in \mathbf{K}$. And let $p(x) \in S_{\text{at}}^1(M)$ be nonalgebraic and p does not split over b for some $b \in M$.

Then $p \upharpoonright b$ has an extension $q(x) \in S_{\text{at}}^1(c)$ such that

$b \in c \in M$ and q is big, but any splitting extension of q is not big.

We may assume that the type q in Proposition 12 above has such property.

We define some closure operator.

Definition 19 Let $M \in \mathbf{K}$ and $p(x) \in S_{\text{at}}^1(M)$. And let p does not split over \emptyset (or some finite parameter) and $p \upharpoonright \emptyset$ is stationary.

The operator cl_p is defined by ;

$cl_p^0(X) = X$ and $cl_p^{n+1}(X) = \{a \in (p \upharpoonright \emptyset)(M) \mid a \notin (p \upharpoonright cl_p^n(X))(M)\}$,
and $cl_p(X) = \bigcup_{n < \omega} cl_p^n(X)$ for any $X \subset (p \upharpoonright \emptyset)(M)$.

We can prove the next fact.

Theorem 20 Let \mathbf{K} has no infinite splitting chain and $M \in \mathbf{K}$ (with $|M| > \aleph_0$).

And let $p(x) \in S_{\text{at}}^1(M)$ be a nonalgebraic type such that p does not split over \emptyset and $p \upharpoonright \emptyset$ has no big splitting extension (or p has a minimal U-rank among such types).

Then $((p \upharpoonright \emptyset)(M), cl_p)$ is pregeometry.

4. Constructible sequence of atomic types

In the argument of categoricity for \ast -excellent AEC, prime models play a crucial role. Now we do not assume the existence of prime models. We try the analogous argument of $F_{\kappa(T)}^a$ -prime models in some large atomic model.

First we check the next lemma.

Lemma 21 (*\mathbf{K} has no infinite splitting chain.*)

Let $M \in \mathbf{K}$. And let $A \subset B \subset M$ and a be such that $\text{tp}_{at}(a/A)$ has a nonsplitting extension over B (or $A \leq_{TV} B$) and $\text{tp}_{at}(a/A)$ is stationary.

Then the following are equivalent ;

- (i) $\text{tp}_{at}(a/A) \vdash \text{tp}_{at}(a/B)$
- (ii) For any a' such that $\text{tp}_{at}(a'/A) = \text{tp}_{at}(a/A)$, $\text{tp}_{at}(a'/B)$ does not split over A .

I define some isolation of atomic types.

Definition 22 Let $a \in M \in \mathbf{K}$ and $A \subset M$.

A type $\text{tp}_{at}(a/A)$ is *quasi-isolated* if there is $b \in M$ such that $\text{tp}_{at}(a/b) \vdash \text{tp}_{at}(a/A)$.

A sequence $\{c_i : i < \alpha\} \subset M$ is *quasi-constructible over A* if, for any $\beta < \alpha$, $\text{tp}_{at}(c_\beta/A \cup \{c_i : i < \beta\})$ is quasi-isolated.

M is *quasi-constructible over A* if $M \setminus A$ can be written as a quasi-constructible sequence.

We can prove the next proposition by using Lemma 21 above.

Proposition 23 Let \mathbf{K} has no infinite splitting chain and $N \in \mathbf{K}$ (with $|N| > \aleph_0$).

And let a nonalgebraic $p(x) \in S_{at}^1(N)$ be such that p does not split over \emptyset and p has no big splitting extension (or p has a minimal U -rank among such types).

(Suppose that $p \upharpoonright \emptyset$ has a Morley sequence I with $|I| > \aleph_0$ in N .)

Then for any basis J of $((p \upharpoonright \emptyset)(N), \text{cl}_p)$, there is a quasi-constructible model over J in N .

5. Categoricity in some large atomic model

At first we recall the definition of Vaughtian triple from [1]. Note that the notion *big* is modified here.

Definition 24 A triple (M, N, ϕ) is called a *Vaughtian triple* if $\phi(M) = \phi(N)$ where $M \prec N \in \mathbf{K}$ with $M \neq N$ and $L(M)$ -formula ϕ is big.

In this chapter, we assume that \mathbf{K} has no infinite splitting chain where \mathbf{K} is an atomic AEC. Under this condition we can prove some results about the two cardinal problem.

I tried the argument of categoricity in this context by means of quasi-constructible model. But I do not have the settled result yet. At present I can prove the next theorem by the properties of generically stable types.

If we try to extend the categoricity result to the whole \mathbf{K} , we need some additional conditions, such as amalgamation property of models, and any atomic set is included in an atomic model, and so on.

In the next Theorem 25, $p \restriction \emptyset$ has a Morley sequence I in N with $|I| = |N|$.

Theorem 25 *Let \mathbf{K} has no infinite splitting chain and $N \in \mathbf{K}$ such that ($|N| > \aleph_0$ and) there is no Vaughtian triple in N .*

And let $p(x) \in S_{\text{at}}^1(N)$ be nonalgebraic such that p does not split over \emptyset and $p \restriction \emptyset$ has no big spitting extension (or p has a minimal U -rank among such types).

Then for $M_i \prec N$ ($i < 2$) with $|M_0| = |M_1|$, $M_0 \cong M_1$.

6. Example of Shelah et al.

Shelah's original work ([4],[5]) showed that categoricity up to \aleph_ω of a sentence in $L_{\omega_1, \omega}$ implies categoricity in all uncountable cardinalities. Shelah and Hart showed the necessity of the assumption by constructing some example ([6]). This example is adapted by Baldwin and Kolesnikov ([1],[2]).

We can not recall the definition of it and details here.

Theorem 26 ([1],[2]) *For each $k < \omega$, there is a $L_{\omega_1, \omega}$ -sentence ϕ_{k+2} such that :*

*ϕ_{k+2} is categorical in μ if $\mu \leq \aleph_k$, and
 ϕ_{k+2} is not categorical in any μ with $\mu > \aleph_k$.*

And they proved the next proposition in [2].

Proposition 27 ([2]) *Let M be the standard model of ϕ_{k+2} of size \aleph_k . Then there are 2^{\aleph_k} Galois types over M .*

This structure is expanded to be an atomic model. And we can check the next fact.

Fact 28 *Let M and ϕ_{k+2} be the $L_{\omega_1, \omega}$ -sentence in the Proposition 27 above. Then M has an infinite splitting chain (in the expanded language).*

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